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Master Class on
Geometry

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Notes on non-Euclidean geometry

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These are notes on hyperbolic geometry, with many digressions on Euclidean and spherical geometry. The treatment of this subject is somehow different from the usual one because it is model-free. This is the way hyperbolic geometry was worked out by Lobachevsky, Bolyai and Gauss, the three founders of the field. The notes nevertheless contain a section on models of hyperbolic geometry (Section 8), and in the introductory part of that section, we point out several advantages of models. For instance, they provide a quick way for doing computations, by using the underlying Euclidean geometry and tools of linear algebra. They also give rise to nice pictures.