LEBESGUE MEASURE AND INTEGRATION

P.K. Jain
Ph.D., F.N.A.Sc.
Emeritus Fellow-UGC
Former Professor and Head
Department of Mathematics
University of Delhi
Delhi, India

V.P. Gupta
Ph.D.
Professor (Retired)
National Council of Education
Research & Training
New Delhi, India

Pankaj Jain
Ph.D.
Associate Professor
Department of Mathematics
Deshbandhu College
(University of Delhi)
New Delhi, India

Anshan
ANSHAN LTD
11a Little Mount Sion
Tunbridge Wells, Kent
TN1 1YS, UK
## CONTENTS

Preface \( v \)

Scheme of Numbering \( ix \)

List of Symbols and Notations \( xi \)

### 1. Preliminaries \( 1-18 \)

1.1 Set and Set Inclusion 1

1.2 Relations 3

1.3 Functions 6

1.4 Axiom of Choice 7

1.5 Supremum and Infimum 8

1.6 Open, Closed and Perfect Sets 9

1.7 Sequences and Series 12

1.8 Continuity and Differentiability 15

### 2. Infinite Sets \( 19-65 \)

2.1 Equivalent Sets 20

2.2 Finite and Infinite Sets 23

2.3 Countable Sets 27

2.4 Uncountable Sets 37

2.5 Cardinality of Sets 42

2.6 Order Relation in Cardinal Numbers 43

2.7 Addition of Cardinal Numbers 46

2.8 Multiplication of Cardinal Numbers 48

2.9 Exponentiation of Cardinal Numbers 49

2.10 Cantor-like Sets 54
CONTENTS

2.11 Cantor Function 61
2.12 Continuum Hypothesis 63

3. Measurable Sets 66–105
3.1 Length of Sets 67
3.2 Outer Measure 68
3.3 Lebesgue Measure 77
3.4 Properties of Measurable Sets 79
3.5 Borel Sets and their Measurability 87
3.6 Further Properties of Measurable Sets 90
3.7 Characterisations of Measurable Sets 96
3.8 Non-measurable Sets 100

4. Measurable Functions 106–146
4.1 Definition 106
4.2 Properties of Measurable Functions 109
4.3 Step Function 112
4.4 Operations on Measurable Functions 113
4.5 Characteristic Function 118
4.6 Simple Function 119
4.7 Continuous Function 120
4.8 Sets of Measure Zero 122
4.9 Borel Measurable Function 125
4.10 Non-Borel Lebesgue Measurable Set 127
4.11 Sequence of Functions 129
4.12 The Structure of Measurable Functions 133
4.13 Littlewood's Three Principles 137
4.14 Convergence in Measure 137

5. Lebesgue Integral 147–193
5.1 Riemann Integral 148
5.2 Lebesgue Integral of a Bounded Function 150
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.3 Comparison of Riemann Integral and Lebesgue Integral</td>
<td>158</td>
</tr>
<tr>
<td>5.4 Properties of the Lebesgue Integral for Bounded Measurable Functions</td>
<td>160</td>
</tr>
<tr>
<td>5.5 Integral of Non-Negative Measurable Functions</td>
<td>166</td>
</tr>
<tr>
<td>5.6 General Lebesgue Integral</td>
<td>177</td>
</tr>
<tr>
<td>5.7 Improper Integrals</td>
<td>189</td>
</tr>
<tr>
<td>6. Differentiation and Integration</td>
<td>194–229</td>
</tr>
<tr>
<td>6.1 Dini Derivatives</td>
<td>194</td>
</tr>
<tr>
<td>6.2 Differentiation of Monotone Functions</td>
<td>199</td>
</tr>
<tr>
<td>6.3 Functions of Bounded Variation</td>
<td>206</td>
</tr>
<tr>
<td>6.4 Differentiation of an Integral</td>
<td>212</td>
</tr>
<tr>
<td>6.5 Lebesgue Set</td>
<td>218</td>
</tr>
<tr>
<td>6.6 Absolutely Continuous Functions</td>
<td>219</td>
</tr>
<tr>
<td>6.7 Integral of the Derivative</td>
<td>226</td>
</tr>
<tr>
<td>7. The Lebesgue $L^p$ Spaces</td>
<td>230–265</td>
</tr>
<tr>
<td>7.1 Notion of Banach Spaces</td>
<td>230</td>
</tr>
<tr>
<td>7.2 The Classes $L^p$</td>
<td>236</td>
</tr>
<tr>
<td>7.3 The Hölder and Minkowski Inequalities</td>
<td>240</td>
</tr>
<tr>
<td>7.4 $L^p$ Banach Spaces</td>
<td>249</td>
</tr>
<tr>
<td>7.5 Convergence in the Mean</td>
<td>253</td>
</tr>
<tr>
<td>7.6 Properties of $L^p$ Spaces</td>
<td>256</td>
</tr>
<tr>
<td>7.7 Bounded Linear Functionals on $L^p$ Spaces</td>
<td>260</td>
</tr>
<tr>
<td>Appendix-I: Existence of Riemann Integral</td>
<td>266–270</td>
</tr>
<tr>
<td>Appendix-II: Nowhere Differentiable Continuous Functions</td>
<td>271–273</td>
</tr>
<tr>
<td>Appendix-III: The Development of the Notion of the Integral</td>
<td>274–290</td>
</tr>
<tr>
<td>Bibliography</td>
<td>291–292</td>
</tr>
<tr>
<td>Index</td>
<td>293–295</td>
</tr>
</tbody>
</table>