Feynman-Kac-Type Theorems and Gibbs Measures on Path Space

With Applications to Rigorous Quantum Field Theory
# Contents

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