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A Practical Course in Differential Equations and Mathematical Modelling
Classical and new methods
Nonlinear mathematical models
Symmetry and invariance principles

Second Edition

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