The Classification of Quasithin Groups

II. Main Theorems: The Classification of Simple QTKE-groups

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Contents of Volumes I and II

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