# Contents

Preface ................................................................. xiii
Authors ................................................................. xvii
Basic Notations ....................................................... xix

## Chapter 1. Statement of Eigenvalue Problems. Basic Methods of Their Solution ................................................. 1

1.1. Statement of the Sturm–Liouville Problem .......................... 1
   1.1.1. Boundary value problem for eigenvalues and eigenfunctions ........................................... 1
   1.1.2. Variational statement of the eigenvalue problem .............................................................. 3

1.2. Analytical Methods of Solving the Sturm–Liouville Problem .................................................. 5
   1.2.1. General scheme of analytical solution .................................................................................... 5
   1.2.2. Reduction to a Fredholm integral equation of the second kind ........................................... 9
   1.2.3. Reduction to a Volterra integral equation of the second kind ............................................. 10

1.3. Solving the Sturm–Liouville Problem by the Method of Regular Perturbations ...................... 12
   1.3.1. Statement of the perturbed problem ..................................................................................... 12
   1.3.2. Standard procedure of asymptotic expansions ..................................................................... 12
   1.3.3. Finding the expansion coefficients ....................................................................................... 13
   1.3.4. Justification questions ......................................................................................................... 14

1.4. Numerical Methods for Solving the Sturm–Liouville Problem ................................................. 14
   1.4.1. The Rayleigh–Ritz method .................................................................................................. 15
   1.4.2. Some general facts and remarks pertaining to other numerical methods in the Sturm–Liouville problem .................................................................................................................. 18

## Chapter 2. The Method of Accelerated Convergence for the Sturm–Liouville Problem ....................... 21

2.1. Numerical-Analytical Upper and Lower Bounds for Eigenvalues ......................................... 21
   2.1.1. The problem of constructing two-sided estimates .................................................................. 21
   2.1.2. Construction and analysis of comparison systems ............................................................... 22

2.2. Criterion of Closeness between the First Eigenvalue and its Upper (Lower) Bound. Introduction of a Small Parameter ................................................................. 23

2.3. Theory of Perturbations ........................................................................................................... 23
   2.3.1. Construction of an equivalent perturbed problem ............................................................... 23
   2.3.2. Approximate solution of the perturbed problem ............................................................... 24
   2.3.3. Reduction of the correction term to differential form .......................................................... 25

2.4. Description of the Method of Accelerated Convergence ....................................................... 26

2.5. Some Applications of the Accelerated Convergence Method ................................................ 27
   2.5.1. Test model problems ......................................................................................................... 27
   2.5.2. A method for the calculation of weighted norms .............................................................. 28
CONTENTS

2.6. The Method of Accelerated Convergence for Higher Eigenvalues 29
  2.6.1. An example with the calculation of two eigenvalues 29
  2.6.2. Some properties of the procedure of finding subsequent eigenvalues 30

2.7. Problems with Boundary Conditions of the Second Kind 31
  2.7.1. Construction of a comparison problem 31
  2.7.2. Approximate solution of the problem 31
  2.7.3. Test problem 31

2.8. Problems with Boundary Conditions of the Third Kind 32
  2.8.1. Statement of the third boundary value problem 32
  2.8.2. Construction of a comparison system 33
  2.8.3. Solution of the perturbed problem 34
  2.8.4. Differential relation between eigenvalues and the interval length 35
  2.8.5. The method of accelerated convergence 35
  2.8.6. Example 36

2.9. Problems with Periodic Boundary Conditions 37
  2.9.1. Statement of the periodic boundary value problem 37
  2.9.2. Main properties of the periodic problem 37
  2.9.3. Construction of upper bounds 38
  2.9.4. Construction of the comparison system 38
  2.9.5. Introduction of a small parameter 39
  2.9.6. Approximate solution of the perturbed problem 40
  2.9.7. The method of accelerated convergence 41
  2.9.8. Examples 43

2.10. Proof of Convergence of Successive Approximations. Existence Theorem 47
  2.10.1. Transformation of the perturbed boundary value problem 47
  2.10.2. Proof of convergence of successive approximations 48

2.11. Proof of Quadratic Convergence 50

2.12. The Method of Hyperaccelerated Convergence 51
  2.12.1. Third-order refinement procedure 51
  2.12.2. An application of the method of hyperaccelerated convergence 52

2.13. Taking into Account Explicit Dependence of Boundary Conditions on Eigenvalues 52

2.14. Exercises 53

Chapter 3. Approximate Analytical Solution of Perturbed Eigenvalue Problems 55

3.1. Statement and Analysis of the Perturbed Sturm–Liouville Problem 55
  3.1.1. Properties of the perturbed spectrum 55
  3.1.2. The problem of secular terms and regularization of the problem 56
  3.1.3. Separation of variables 57

3.2. Approximate Solution of the Boundary Value Problem 58
  3.2.1. Construction of eigenfrequencies and phases of partial vibrations 58
  3.2.2. Finding eigenfunctions and the construction of an orthonormal basis 60
  3.2.3. Remarks 61

3.3. Approximation of Functions in Terms of the Approximate Basis 61
  3.3.1. The problem of expansion in terms of an approximate basis 61
  3.3.2. Uniform estimates 63
5.4. Finding Amplitudes and Shapes of Free Vibrations ........................................... 85
  5.4.1. Approximate calculation of higher mode amplitudes .................................. 85
  5.4.2. Finding eigenfunctions corresponding to higher modes ............................. 86

5.5. Other Types of Boundary Value Problems .......................................................... 87
  5.5.1. Boundary conditions of the second kind ...................................................... 87
  5.5.2. General boundary conditions of the third kind ........................................... 87
  5.5.3. Remarks about generalizations ..................................................................... 88

5.6. Calculations for Some Specific Mechanical Systems ........................................... 88
  5.6.1. Longitudinal vibrations of an inhomogeneous rectilinear beam .................... 88
  5.6.2. Vibrations of an inhomogeneous string ....................................................... 89
  5.6.3. Asymptotics of eigenvalues of the Hill problem ....................................... 90
  5.6.4. Spatial vibrations of a satellite .................................................................... 91

5.7. Exercises ........................................................................................................... 93

Properties .................................................................................................................. 95

6.1. Statement of a Self-Conjugate Fourth-Order Boundary Value Problem ............... 95
  6.1.1. Statement of the problem in differential form. Some remarks ........................ 95
  6.1.2. Statement of the problem in variational form ............................................... 96
  6.1.3. Introduction of natural physical variables .................................................... 97
  6.1.4. Scheme of solution ....................................................................................... 97

6.2. The Method of Sagittary Function. Sturm's Theorems ......................................... 99
  6.2.1. Construction of the characteristic equation and the sagittary function .......... 99
  6.2.2. Oscillation properties of the sagittary function ........................................... 100

6.3. Computation Algorithms of the Shooting Method Based on the Sagittary Function .. 102
  6.3.1. Algorithm of shooting with respect to the ordinate ...................................... 102
  6.3.2. Algorithm of shooting with respect to the abscissa ...................................... 103

6.4. Examples ........................................................................................................... 104
  6.4.1. A model test example .................................................................................... 105
  6.4.2. Comparison with the results of S. Gould ...................................................... 106
  6.4.3. Parametric synthesis for conical beams ....................................................... 106

Chapter 7. The Method of Accelerated Convergence for Eigenvalue Problems for
Fourth-Order Equations ............................................................................................. 109

7.1. Two-Sided Estimates for Lower Mode Eigenvalues ........................................... 109
  7.1.1. Differential and variational statements of the problem ............................... 109
  7.1.2. Construction of upper bounds ...................................................................... 110
  7.1.3. Relation between the upper bound and the length of the interval ................. 111
  7.1.4. Construction of lower bounds and two-sided estimates ............................. 113

7.2. Closeness Criterion and Perturbation Theory ..................................................... 113
  7.2.1. Introduction of a small parameter ............................................................... 113
  7.2.2. An approximate solution of the perturbed problem .................................... 114

7.3. The Method of Accelerated Convergence for Fourth-Order Boundary Value Problems 115
  7.3.1. A differential relation between the eigenvalue and the length of the interval .. 115
  7.3.2. Algorithm of the accelerated convergence method .................................... 115

7.4. Other Types of Boundary Conditions .................................................................. 116
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.5.</td>
<td>Procedure of Continuation in a Parameter</td>
<td>117</td>
</tr>
<tr>
<td>7.6.</td>
<td>Model Problems</td>
<td>117</td>
</tr>
<tr>
<td>7.6.1.</td>
<td>General remarks about calculations</td>
<td>117</td>
</tr>
<tr>
<td>7.6.2.</td>
<td>Test examples with analytically integrable equations</td>
<td>118</td>
</tr>
<tr>
<td>7.6.3.</td>
<td>Problem of transverse vibrations of an inhomogeneous beam occurring in applications</td>
<td>119</td>
</tr>
<tr>
<td>Chapter 8.</td>
<td>Perturbation Method in Eigenvalue Problems for Fourth-Order Equations</td>
<td>121</td>
</tr>
<tr>
<td>8.1.</td>
<td>Reduction of the Original Problem to the Standard Perturbed Boundary Value Problem</td>
<td>121</td>
</tr>
<tr>
<td>8.1.1.</td>
<td>Statement of the initial boundary value problem; preliminary remarks</td>
<td>121</td>
</tr>
<tr>
<td>8.1.2.</td>
<td>Reduction to perturbed boundary value problems</td>
<td>123</td>
</tr>
<tr>
<td>8.1.3.</td>
<td>Some features of the standard procedure of the perturbation method</td>
<td>123</td>
</tr>
<tr>
<td>8.2.</td>
<td>Regularization of the Perturbation Method</td>
<td>124</td>
</tr>
<tr>
<td>8.2.1.</td>
<td>Transformation of the independent variable</td>
<td>124</td>
</tr>
<tr>
<td>8.2.2.</td>
<td>Regular procedure of the perturbation method</td>
<td>125</td>
</tr>
<tr>
<td>8.2.3.</td>
<td>Justification of the perturbation method</td>
<td>126</td>
</tr>
<tr>
<td>8.3.</td>
<td>Motion Control Problem</td>
<td>128</td>
</tr>
<tr>
<td>8.4.</td>
<td>Finding the Eigenvalues and the Eigenfunctions in the First Approximation</td>
<td>128</td>
</tr>
<tr>
<td>8.5.</td>
<td>Exercises</td>
<td>131</td>
</tr>
<tr>
<td>Chapter 9.</td>
<td>Sturm–Liouville Problems for Vector-Valued Functions</td>
<td>133</td>
</tr>
<tr>
<td>9.1.</td>
<td>Setting of the Problem. Preliminary Remarks</td>
<td>133</td>
</tr>
<tr>
<td>9.1.1.</td>
<td>Statement of the problem in differential form</td>
<td>133</td>
</tr>
<tr>
<td>9.1.2.</td>
<td>Variational statement of the problem</td>
<td>133</td>
</tr>
<tr>
<td>9.2.</td>
<td>Closeness Criterion and Perturbation Theory</td>
<td>134</td>
</tr>
<tr>
<td>9.2.1.</td>
<td>Construction of the comparison problem; analysis of its properties</td>
<td>134</td>
</tr>
<tr>
<td>9.2.2.</td>
<td>Introduction of a small parameter</td>
<td>135</td>
</tr>
<tr>
<td>9.2.3.</td>
<td>Approximate solution of the problem</td>
<td>135</td>
</tr>
<tr>
<td>9.3.</td>
<td>The Method of Accelerated Convergence for the Sturm–Liouville Problem for Vector-Valued Functions</td>
<td>136</td>
</tr>
<tr>
<td>9.3.1.</td>
<td>Properties of the first approximation of the solution</td>
<td>136</td>
</tr>
<tr>
<td>9.3.2.</td>
<td>Algorithm of accelerated convergence for vector problems</td>
<td>137</td>
</tr>
<tr>
<td>9.4.</td>
<td>Model Problems</td>
<td>138</td>
</tr>
<tr>
<td>9.4.1.</td>
<td>A system of Euler type</td>
<td>138</td>
</tr>
<tr>
<td>9.4.2.</td>
<td>A system with periodic coefficients</td>
<td>139</td>
</tr>
<tr>
<td>9.5.</td>
<td>Exercises</td>
<td>139</td>
</tr>
<tr>
<td>Chapter 10.</td>
<td>Vibrations and Stability of Elastic Systems</td>
<td>141</td>
</tr>
<tr>
<td>10.1.</td>
<td>Plane Vibrations of a Rotating Heavy Thread and Their Stability</td>
<td>141</td>
</tr>
<tr>
<td>10.1.1.</td>
<td>Statement of the initial boundary value problem. Its solution by the Fourier method</td>
<td>141</td>
</tr>
<tr>
<td>10.1.2.</td>
<td>Free vibrations of a rotating heavy homogeneous string subjected to tension</td>
<td>145</td>
</tr>
<tr>
<td>10.1.3.</td>
<td>Vibrations of an inhomogeneous thread</td>
<td>149</td>
</tr>
<tr>
<td>10.2.</td>
<td>Parametric Synthesis in the Problem of Instability of an Inhomogeneous Beam</td>
<td>152</td>
</tr>
<tr>
<td>10.2.1.</td>
<td>Setting of the problem of longitudinal bending of an elastic beam</td>
<td>152</td>
</tr>
<tr>
<td>10.2.2.</td>
<td>Calculation of the critical force for some rigidity distributions</td>
<td>154</td>
</tr>
</tbody>
</table>
13.4. Some Generalizations .................................................. 194
  13.4.1. Perturbation of the surface density function .................. 194
  13.4.2. Nonuniform membrane tension .................................. 195
  13.4.3. The presence of elastic environment .......................... 195
  13.4.4. Taking into account perturbations of general form ........... 196

13.5. Examples ............................................................... 196
  13.5.1. Inhomogeneity with respect to one coordinate ............... 196
  13.5.2. Symmetric inhomogeneity ........................................ 197
  13.5.3. Multicoordinate approximation .................................. 198

13.6. Exercises .............................................................. 200

Chapter 14. Free Vibrations of a Rectangular Membrane with Sharply Varying Surface
Density ................................................................. 201

14.1. Statement of the Problem of Free Vibrations of an Inhomogeneous Rectangular
Membrane ................................................................. 201
  14.1.1. Preliminary remarks .............................................. 201
  14.1.2. Statement of the boundary value problem ...................... 202

14.2. Construction of the Generating Solution .......................... 203
  14.2.1. Separation of the variables in the unperturbed problem ...... 203
  14.2.2. A scheme for the construction of the generating solution ... 203

14.3. Membrane Model with Sharply Changing Surface Density ......... 204
  14.3.1. Approximation of the density function ......................... 204
  14.3.2. Brief description of the algorithm .............................. 205
  14.3.3. Software ........................................................ 207

14.4. Calculation Results and Conclusions ................................ 208
  14.4.1. Calculation results for the symmetrical cross ............... 209
  14.4.2. Calculation results of the shifted cross ...................... 210
  14.4.3. Calculation results for the nonsymmetric cross .............. 211
  14.4.4. Conclusions ................................................... 212

Chapter 15. Free Vibrations of Elastic Systems in Elliptic Domains .... 215

15.1. Free Vibrations of a Homogeneous Elliptic Membrane ............ 215
  15.1.1. Preliminary remarks regarding the present state of the investigations .. 215
  15.1.2. Setting of the problem ......................................... 216
  15.1.3. Variational approach and the construction of highly precise estimates .... 217
  15.1.4. Construction of approximate analytical expressions for eigenvalues of elliptic
membranes with small eccentricity ..................................... 222
  15.1.5. Asymptotic expansions of eigenvalues for large eccentricity values .... 223
  15.1.6. Finding eigenfrequencies and vibration shapes of an elliptic membrane by the
method of accelerated convergence .................................. 225
  15.1.7. Conclusions ................................................... 226

15.2. Free Vibrations of an Elliptic Plate with Clamped Edge ........ 227
  15.2.1. Preliminary remarks ............................................ 227
  15.2.2. Setting of the problem ......................................... 227
  15.2.3. Estimates for the frequency of the lowest vibration mode with the help of an
elliptically symmetrical test function .............................. 228