Introduction to Modern Analysis

Shmuel Kantorovitz
Bar Ilan University,
Ramat Gan, Israel
Contents

1 Measures 1
1.1 Measurable sets and functions 1
1.2 Positive measures 7
1.3 Integration of non-negative measurable functions 9
1.4 Integrable functions 15
1.5 $L^p$-spaces 22
1.6 Inner product 29
1.7 Hilbert space: a first look 32
1.8 The Lebesgue–Radon–Nikodym theorem 34
1.9 Complex measures 39
1.10 Convergence 46
1.11 Convergence on finite measure space 49
1.12 Distribution function 50
1.13 Truncation 52
Exercises 54

2 Construction of measures 57
2.1 Semi-algebras 57
2.2 Outer measures 59
2.3 Extension of measures on algebras 62
2.4 Structure of measurable sets 63
2.5 Construction of Lebesgue–Stieltjes measures 64
2.6 Riemann versus Lebesgue 67
2.7 Product measure 69
Exercises 73

3 Measure and topology 77
3.1 Partition of unity 77
3.2 Positive linear functionals 79
3.3 The Riesz–Markov representation theorem 87
3.4 Lusin’s theorem 89
3.5 The support of a measure 92
3.6 Measures on $\mathbb{R}^k$; differentiability 93
Exercises 97
4 Continuous linear functionals 102
4.1 Linear maps 102
4.2 The conjugates of Lebesgue spaces 104
4.3 The conjugate of $C_c(X)$ 109
4.4 The Riesz representation theorem 111
4.5 Haar measure 113
Exercises 121

5 Duality 123
5.1 The Hahn–Banach theorem 123
5.2 Reflexivity 127
5.3 Separation 130
5.4 Topological vector spaces 133
5.5 Weak topologies 135
5.6 Extremal points 139
5.7 The Stone–Weierstrass theorem 143
5.8 Operators between Lebesgue spaces: Marcinkiewicz’s interpolation theorem 145
Exercises 150

6 Bounded operators 153
6.1 Category 153
6.2 The uniform boundedness theorem 154
6.3 The open mapping theorem 156
6.4 Graphs 159
6.5 Quotient space 160
6.6 Operator topologies 161
Exercises 164

7 Banach algebras 170
7.1 Basics 170
7.2 Commutative Banach algebras 178
7.3 Involution 181
7.4 Normal elements 183
7.5 General $B^*$-algebras 185
7.6 The Gelfand–Naimark–Segal construction 190
Exercises 195

8 Hilbert spaces 203
8.1 Orthonormal sets 203
8.2 Projections 206
8.3 Orthonormal bases 208
8.4 Hilbert dimension 211
8.5 Isomorphism of Hilbert spaces 212
8.6 Canonical model 213
8.7 Commutants 214
Exercises 215
Contents

9 Integral representation 223
9.1 Spectral measure on a Banach subspace 223
9.2 Integration 224
9.3 Case $Z = X$ 226
9.4 The spectral theorem for normal operators 229
9.5 Parts of the spectrum 231
9.6 Spectral representation 233
9.7 Renorming method 235
9.8 Semi-simplicity space 237
9.9 Resolution of the identity on $Z$ 239
9.10 Analytic operational calculus 243
9.11 Isolated points of the spectrum 246
9.12 Compact operators 248
Exercises 252

10 Unbounded operators 258
10.1 Basics 258
10.2 The Hilbert adjoint 261
10.3 The spectral theorem for unbounded selfadjoint operators 264
10.4 The operational calculus for unbounded selfadjoint operators 265
10.5 The semi-simplicity space for unbounded operators in Banach space 267
10.6 Symmetric operators in Hilbert space 271
Exercises 275

Application I Probability 283
I.1 Heuristics 283
I.2 Probability space 285
I.3 Probability distributions 298
I.4 Characteristic functions 307
I.5 Vector-valued random variables 315
I.6 Estimation and decision 324
I.7 Conditional probability 336
I.8 Series of $L^2$ random variables 349
I.9 Infinite divisibility 355
I.10 More on sequences of random variables 359

Application II Distributions 364
II.1 Preliminaries 364
II.2 Distributions 366
II.3 Temperate distributions 376
II.4 Fundamental solutions 392
II.5 Solution in $\mathcal{E}'$ 396
II.6 Regularity of solutions 398
II.7  Variable coefficients  400
II.8  Convolution operators  404
II.9  Some holomorphic semigroups  415

Bibliography  421

Index  425