# Contents

Introduction 1

1 Differential forms 9
   1.1 Usual notation ........................................... 9
   1.2 Complex differential forms ............................ 10
   1.3 Operations on complex differential forms .......... 11
   1.4 Integration with respect to a part of variables .... 14
   1.5 The differential form $|F|$ ............................ 15
   1.6 More spaces of differential forms ................. 16

2 Differential forms with coefficients in $2 \times 2$-matrices 19
   2.1 Classes $G_p(\Omega), G_p(\Omega)$ ...................... 19
   2.2 Matrix-valued differential forms ..................... 19
   2.3 The hyperholomorphic Cauchy-Riemann operators .... 21
       on $\mathcal{G}_1$ and $\mathcal{G}_1$ ........................... 21
   2.4 Formula for $d\left(F \wedge G\right)$ ................... 24
   2.5 Differential matrix forms of the unit normal ...... 24
   2.6 Formula for $d_c\left(F \wedge \overline{\sigma} \wedge G\right)$ ......... 28
   2.7 Exterior differentiation and the hyperholomorphic
       Cauchy-Riemann operators ............................ 32
   2.8 Stokes formula compatible with the hyperholo-
       morphic Cauchy-Riemann operators .................... 32
   2.9 The Cauchy kernel for the null-sets of the hyperholo-
       morphic Cauchy-Riemann operators .................... 34
   2.10 Structure of the product $\mathcal{K}_\mathcal{D} \wedge \overline{\sigma}$ .... 35
   2.11 Borel-Pompeiu (or Cauchy-Green) formula for
       smooth differential matrix-forms .................... 39

v
2.11.1 Structure of the Borel-Pompeiu formula 44
2.11.2 The case $m = 1$ 47
2.11.3 The case $m = 2$ 48
2.11.4 Notations for some integrals in $\mathbb{C}^2$ 51
2.11.5 Formulas of the Borel-Pompeiu type in $\mathbb{C}^2$ 54
2.11.6 Complements to the Borel-Pompeiu-type formulas in $\mathbb{C}^2$ 55
2.11.7 The case $m > 2$ 55
2.11.8 Notations for some integrals in $\mathbb{C}^m$ 57
2.11.9 Formulas of the Borel-Pompeiu type in $\mathbb{C}^m$ 58
2.11.10 Complements to the Borel-Pompeiu-type formulas in $\mathbb{C}^m$ 58

3 Hyperholomorphic functions and differential forms in $\mathbb{C}^m$ 61
  3.1 Hyperholomorphy in $\mathbb{C}^m$ 61
  3.2 Hyperholomorphy in one variable 62
  3.3 Hyperholomorphy in two variables 63
  3.4 Hyperholomorphy in three variables 65
  3.5 Hyperholomorphy for any number of variables 70
  3.6 Observation about right-hand-side hyperholomorphy 73

4 Hyperholomorphic Cauchy's integral theorems 75
  4.1 The Cauchy integral theorem for left-hyperholomorphic matrix-valued differential forms 75
  4.2 The Cauchy integral theorem for right-G-hyperholomorphic m.v.d.f. 75
  4.3 Some auxiliary computations 76
  4.4 More auxiliary computations 77
  4.5 The Cauchy integral theorem for holomorphic functions of several complex variables 78
  4.6 The Cauchy integral theorem for antiholomorphic functions of several complex variables 78
  4.7 The Cauchy integral theorem for functions holomorphic in some variables and antiholomorphic in the rest of variables 79
  4.8 Concluding remarks 80
5 Hyperholomorphic Morera's theorems
5.1 Left-hyperholomorphic Morera theorem ............... 81
5.2 Version of a right-hyperholomorphic Morera theorem 82
5.3 Morera's theorem for holomorphic functions of
several complex variables ............................ 84
5.4 Morera's theorem for antiholomorphic functions of
several complex variables ............................ 85
5.5 The Morera theorem for functions holomorphic in some
variables and antiholomorphic in the rest of variables 86

6 Hyperholomorphic Cauchy's integral representations
6.1 Cauchy's integral representation for left-
hyperholomorphic matrix-valued differential forms . 89
6.2 A consequence for holomorphic functions ............ 90
6.3 A consequence for antiholomorphic functions ........ 90
6.4 A consequence for holomorphic-like functions ....... 91
6.5 Bochner-Martinelli integral representation for holo-
morphic functions of several complex variables, and
hyperholomorphic function theory ....................... 92
6.6 Bochner-Martinelli integral representation for antiholo-
morphic functions of several complex variables, and
hyperholomorphic function theory ....................... 92
6.7 Bochner-Martinelli integral representation for func-
tions holomorphic in some variables and antiholo-
morphic in the rest, and hyperholomorphic function
theory .................................................. 93

7 Hyperholomorphic $\overline{D}$-problem
7.1 Some reasonings from one variable theory ............ 95
7.2 Right inverse operators to the hyperholomorphic
Cauchy-Riemann operators .............................. 97
7.2.1 Structure of the formula of Theorem 7.2 .......... 99
7.2.2 Case $m = 1$ ........................................ 101
7.2.3 Case $m = 2$ ........................................ 102
7.2.4 Case $m > 2$ ........................................ 106
7.2.5 Analogs of (7.1.7) .................................... 109
7.2.6 Commutativity relations for T-type operators .... 110
7.3 Solution of the hyperholomorphic $\overline{D}$-problem .... 110
### 7.4 Structure of the general solution of the hyperholomorphic $\overline{D}$-problem

111

### 7.5 $\overline{D}$-type problem for the Hodge-Dirac operator

114

### 8 Complex Hodge-Dolbeault system, the $\bar{\partial}$-problem and the Koppelman formula

117

#### 8.1 Definition of the complex Hodge-Dolbeault system

117

#### 8.2 Relation with hyperholomorphic case

118

#### 8.3 The Cauchy integral theorem for solutions of degree $p$ for the complex Hodge-Dolbeault system

119

#### 8.4 The Cauchy integral theorem for arbitrary solutions of the complex Hodge-Dolbeault system

121

#### 8.5 Morera's theorem for solutions of degree $p$ for the complex Hodge-Dolbeault system

122

#### 8.6 Morera's theorem for arbitrary solutions of the complex Hodge-Dolbeault system

123

#### 8.7 Solutions of a fixed degree

124

#### 8.8 Arbitrary solutions

124

#### 8.9 Bochner-Martinelli-type integral representation for solutions of degree $s$ of the complex Hodge-Dolbeault system

125

#### 8.10 Bochner-Martinelli-type integral representation for arbitrary solutions of the complex Hodge-Dolbeault system

126

#### 8.11 Solution of the $\bar{\partial}$-type problem for the complex Hodge-Dolbeault system in a bounded domain in $\mathbb{C}^m$

127

#### 8.12 Complex $\bar{\partial}$-problem and the $\bar{\partial}$-type problem for the complex Hodge-Dolbeault system

128

#### 8.13 $\bar{\partial}$-problem for differential forms

130

8.13.1 $\bar{\partial}$-problem for functions of several complex variables

131

#### 8.14 General situation of the Borel-Pompeiu representation

132

#### 8.15 Partial derivatives of integrals with a weak singularity

138

#### 8.16 Theorem 8.15 in $\mathbb{C}^2$

140

#### 8.17 Formula (8.14.3) in $\mathbb{C}^2$

141
8.18 Integral representation (8.14.3) for a (0, 1)-differential form in $\mathbb{C}^2$, in terms of its coefficients ................................................................. 143
8.19 Koppelman's formula in $\mathbb{C}^2$ ................................................. 143
8.20 Koppelman's formula in $\mathbb{C}^2$ for a (0, 1) - differential form, in terms of its coefficients ................................................................. 144
8.21 Comparison of Propositions 8.18 and 8.20 ............................... 145
8.22 Koppelman's formula in $\mathbb{C}^2$ and hyperholomorphic theory ................................................................. 147
8.23 Definition of $\rho_{H,K}$ ................................................................. 147
8.24 A reformulation of the Borel-Pompeiu formula ................................................................. 148
8.25 Identity (8.14.4) for a d.f. of a fixed degree ............................... 151
8.26 About the Koppelman formula ................................................................. 153
8.27 Auxiliary computations ................................................................. 159
8.28 The Koppelman formula for solutions of the complex Hodge-Dolbeault system ................................................................. 162
8.29 Appendix: properties of $\rho_{H,K}$ ................................................................. 163

9 Hyperholomorphic theory and Clifford analysis ............................... 167
9.1 One way to introduce a complex Clifford algebra ................................................................. 167
9.1.1 Classical definition of a complex Clifford algebra ................................................................. 168
9.2 Some differential operators on $\mathbb{W}_m$-valued functions ................................................................. 170
9.2.1 Factorization of the Laplace operator ................................................................. 171
9.3 Relation of the operators $\bar{\partial}$ and $\bar{\partial}^\wedge$ with the Dirac operator of Clifford analysis ................................................................. 173
9.4 Matrix algebra with entries from $\mathbb{W}_m$ ................................................................. 174
9.5 The matrix Dirac operators ................................................................. 175
9.5.1 Factorization of the Laplace operator on $\mathbb{W}_m$-valued functions ................................................................. 176
9.6 The fundamental solution of the matrix Dirac operators ................................................................. 177
9.7 Borel-Pompeiu formulas for $\mathbb{W}_m$-valued functions ................................................................. 179