THE KURZWEIL-HENSTOCK INTEGRAL AND ITS DIFFERENTIALS

A Unified Theory of Integration on $\mathbb{R}$ and $\mathbb{R}^n$

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