THE PRINCIPLES OF NEWTONIAN AND QUANTUM MECHANICS

The Need for Planck's Constant, $h$

M A de Gosson

Blekinge Institute of Technology, Sweden

Imperial College Press
1.9 Interpretations
   1.9.1 Epistemology or Ontology?
   1.9.2 The Copenhagen Interpretation
   1.9.3 The Bohmian Interpretation
   1.9.4 The Platonic Point of View

2 NEWTONIAN MECHANICS
   2.1 Maxwell’s Principle and the Lagrange Form
      2.1.1 The Hamilton Vector Field
      2.1.2 Force Fields
      2.1.3 Statement of Maxwell’s Principle
      2.1.4 Magnetic Monopoles and the Dirac String
      2.1.5 The Lagrange Form
      2.1.6 N-Particle Systems
   2.2 Hamilton’s Equations
      2.2.1 The Poincaré-Cartan Form and Hamilton’s Equations
      2.2.2 Hamiltonians for N-Particle Systems
      2.2.3 The Transformation Law for Hamilton Vector Fields
      2.2.4 The Suspended Hamiltonian Vector Field
   2.3 Galilean Covariance
      2.3.1 Inertial Frames
      2.3.2 The Galilean Group Gal(3)
      2.3.3 Galilean Covariance of Hamilton’s Equations
   2.4 Constants of the Motion and Integrable Systems
      2.4.1 The Poisson Bracket
      2.4.2 Constants of the Motion and Liouville’s Equation
      2.4.3 Constants of the Motion in Involution
   2.5 Liouville’s Equation and Statistical Mechanics
      2.5.1 Liouville’s Condition
      2.5.2 Marginal Probabilities
      2.5.3 Distributional Densities: An Example

3 THE SYMPLECTIC GROUP
   3.1 Symplectic Matrices and $Sp(n)$
   3.2 Symplectic Invariance of Hamiltonian Flows
      3.2.1 Notations and Terminology
      3.2.2 Proof of the Symplectic Invariance of Hamiltonian Flows
      3.2.3 Another Proof of the Symplectic Invariance of Flows*
   3.3 The Properties of $Sp(n)$
### 3.3 Subgroups, Lie Algebra, and Lie Group

- **3.3.1 The Subgroups** $U(n)$ and $O(n)$ of $Sp(n)$
- **3.3.2 The Lie Algebra** $sp(n)$
- **3.3.3 $Sp(n)$ as a Lie Group**

### 3.4 Quadratic Hamiltonians

- **3.4.1 The Linear Symmetric Triatomic Molecule**
- **3.4.2 Electron in a Uniform Magnetic Field**

### 3.5 Inhomogeneous Symplectic Group

- **3.5.1 Galilean Transformations and $ISp(n)$**

### 3.6 An Illuminating Analogy

- **3.6.1 The Optical Hamiltonian**
- **3.6.2 Paraxial Optics**

### 3.7 Gromov's Non-Squeezing Theorem

- **3.7.1 Liouville's Theorem Revisited**
- **3.7.2 Gromov's Theorem**
- **3.7.3 The Uncertainty Principle in Classical Mechanics**

### 3.8 Symplectic Capacity and Periodic Orbits

- **3.8.1 The Capacity of an Ellipsoid**
- **3.8.2 Symplectic Area and Volume**

### 3.9 Capacity and Periodic Orbits

- **3.9.1 Periodic Hamiltonian Orbits**
- **3.9.2 Action of Periodic Orbits and Capacity**

### 3.10 Cell Quantization of Phase Space

- **3.10.1 Stationary States of Schrödinger's Equation**
- **3.10.2 Quantum Cells and the Minimum Capacity Principle**
- **3.10.3 Quantization of the $N$-Dimensional Harmonic Oscillator**

### 4 Action and Phase

- **4.1 Introduction**
- **4.2 The Fundamental Property of the Poincaré-Cartan Form**
  - **4.2.1 Helmholtz's Theorem: The Case $n = 1$**
  - **4.2.2 Helmholtz's Theorem: The General Case**
- **4.3 Free Symplectomorphisms and Generating Functions**
  - **4.3.1 Generating Functions**
  - **4.3.2 Optical Analogy: The Eikonal**
- **4.4 Generating Functions and Action**
  - **4.4.1 The Generating Function Determined by $H$**
  - **4.4.2 Action vs. Generating Function**
  - **4.4.3 Gauge Transformations and Generating Functions**
CONTENTS

4.4.4 Solving Hamilton’s Equations with W
4.4.5 The Cauchy Problem for Hamilton-Jacobi’s Equation

4.5 Short-Time Approximations to the Action
  4.5.1 The Case of a Scalar Potential
  4.5.2 One Particle in a Gauge (A, U)
  4.5.3 Many-Particle Systems in a Gauge (A, U)

4.6 Lagrangian Manifolds
  4.6.1 Definitions and Basic Properties
  4.6.2 Lagrangian Manifolds in Mechanics

4.7 The Phase of a Lagrangian Manifold
  4.7.1 The Phase of an Exact Lagrangian Manifold
  4.7.2 The Universal Covering of a Manifold*
  4.7.3 The Phase: General Case
  4.7.4 Phase and Hamiltonian Motion

4.8 Keller-Maslov Quantization
  4.8.1 The Maslov Index for Loops
  4.8.2 Quantization of Lagrangian Manifolds
  4.8.3 Illustration: The Plane Rotator

5 SEMI-CLASSICAL MECHANICS

5.1 Bohmian Motion and Half-Densities
  5.1.1 Wave-Forms on Exact Lagrangian Manifolds
  5.1.2 Semi-Classical Mechanics
  5.1.3 Wave-Forms: Introductory Example

5.2 The Leray Index and the Signature Function*
  5.2.1 Cohomological Notations
  5.2.2 The Leray Index: n = 1
  5.2.3 The Leray Index: General Case
  5.2.4 Properties of the Leray Index
  5.2.5 More on the Signature Function
  5.2.6 The Reduced Leray Index

5.3 De Rham Forms
  5.3.1 Volumes and their Absolute Values
  5.3.2 Construction of De Rham Forms on Manifolds
  5.3.3 De Rham Forms on Lagrangian Manifolds

5.4 Wave-Forms on a Lagrangian Manifold
  5.4.1 Definition of Wave Forms
  5.4.2 The Classical Motion of Wave-Forms
### Contents

5.4.3. The Shadow of a Wave-Form

#### 6 THE METAPLECTIC GROUP AND THE MASLOV INDEX

6.1 Introduction
- 6.1.1 Could Schrödinger have Done it Rigorously?
- 6.1.2 Schrödinger’s Idea
- 6.1.3 $Sp(n)$’s “Big Brother” $Mp(n)$

6.2 Free Symplectic Matrices and their Generating Functions
- 6.2.1 Free Symplectic Matrices
- 6.2.2 The Case of Affine Symplectomorphisms
- 6.2.3 The Generators of $Sp(n)$

6.3 The Metaplectic Group $Mp(n)$
- 6.3.1 Quadratic Fourier Transforms
- 6.3.2 The Operators $ML,m$ and $VP$

6.4 The Projections $\Pi$ and $\Pi^c$
- 6.4.1 Construction of the Projection $\Pi$
- 6.4.2 The Covering Groups $Mp^c(n)$

6.5 The Maslov Index on $Mp(n)$
- 6.5.1 Maslov Index: A “Simple” Example
- 6.5.2 Definition of the Maslov Index on $Mp(n)$

6.6 The Cohomological Meaning of the Maslov Index*
- 6.6.1 Group Cocycles on $Sp(n)$
- 6.6.2 The Fundamental Property of $m(\cdot)$

6.7 The Inhomogeneous Metaplectic Group
- 6.7.1 The Heisenberg Group
- 6.7.2 The Group $IMP(n)$

6.8 The Metaplectic Group and Wave Optics
- 6.8.1 The Passage from Geometric to Wave Optics

6.9 The Groups $Symp(n)$ and $Ham(n)$*
- 6.9.1 A Topological Property of $Symp(n)$
- 6.9.2 The Group $Ham(n)$ of Hamiltonian Symplectomorphisms
- 6.9.3 The Groenewold-Van Hove Theorem

#### 7 SCHRODINGER’S EQUATION AND THE METATRON

7.1 Schrödinger’s Equation for the Free Particle
- 7.1.1 The Free Particle’s Phase
- 7.1.2 The Free Particle Propagator
- 7.1.3 An Explicit Expression for $G$
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.1.4 The Metaplectic Representation of the Free Flow</td>
<td></td>
</tr>
<tr>
<td>7.1.5 More Quadratic Hamiltonians</td>
<td></td>
</tr>
<tr>
<td>7.2 Van Vleck's Determinant</td>
<td>277</td>
</tr>
<tr>
<td>7.2.1 Trajectory Densities</td>
<td></td>
</tr>
<tr>
<td>7.3 The Continuity Equation for Van Vleck's Density</td>
<td>280</td>
</tr>
<tr>
<td>7.3.1 A Property of Differential Systems</td>
<td></td>
</tr>
<tr>
<td>7.3.2 The Continuity Equation for Van Vleck's Density</td>
<td></td>
</tr>
<tr>
<td>7.4 The Short-Time Propagator</td>
<td>284</td>
</tr>
<tr>
<td>7.4.1 Properties of the Short-Time Propagator</td>
<td></td>
</tr>
<tr>
<td>7.5 The Case of Quadratic Hamiltonians</td>
<td>288</td>
</tr>
<tr>
<td>7.5.1 Exact Green Function</td>
<td></td>
</tr>
<tr>
<td>7.5.2 Exact Solutions of Schrödinger's Equation</td>
<td></td>
</tr>
<tr>
<td>7.6 Solving Schrödinger's Equation: General Case</td>
<td>290</td>
</tr>
<tr>
<td>7.6.1 The Short-Time Propagator and Causality</td>
<td></td>
</tr>
<tr>
<td>7.6.2 Statement of the Main Theorem</td>
<td></td>
</tr>
<tr>
<td>7.6.3 The Formula of Stationary Phase</td>
<td></td>
</tr>
<tr>
<td>7.6.4 Two Lemmas — and the Proof</td>
<td></td>
</tr>
<tr>
<td>7.7 Metatrons and the Implicate Order</td>
<td>300</td>
</tr>
<tr>
<td>7.7.1 Unfolding and Implicate Order</td>
<td></td>
</tr>
<tr>
<td>7.7.2 Prediction and Retrodiction</td>
<td></td>
</tr>
<tr>
<td>7.7.3 The Lie-Trotter Formula for Flows</td>
<td></td>
</tr>
<tr>
<td>7.7.4 The &quot;Unfolded&quot; Metatron</td>
<td></td>
</tr>
<tr>
<td>7.7.5 The Generalized Metaplectic Representation</td>
<td></td>
</tr>
<tr>
<td>7.8 Phase Space and Schrödinger's Equation</td>
<td>313</td>
</tr>
<tr>
<td>7.8.1 Phase Space and Quantum Mechanics</td>
<td></td>
</tr>
<tr>
<td>7.8.2 Mixed Representations in Quantum Mechanics</td>
<td></td>
</tr>
<tr>
<td>7.8.3 Complementarity and the Implicate Order</td>
<td></td>
</tr>
<tr>
<td>A Symplectic Linear Algebra</td>
<td>323</td>
</tr>
<tr>
<td>B The Lie-Trotter Formula for Flows</td>
<td>327</td>
</tr>
<tr>
<td>C The Heisenberg Groups</td>
<td>331</td>
</tr>
<tr>
<td>D The Bundle of s-Densities</td>
<td>335</td>
</tr>
<tr>
<td>E The Lagrangian Grassmannian</td>
<td>339</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>343</td>
</tr>
<tr>
<td>INDEX</td>
<td>353</td>
</tr>
</tbody>
</table>