## Contents

Preface xi  

Chapter 1. Introduction 1  

Chapter 2. Generalities on $G/B$ and $G/Q$ 7  
2.1. Abstract root systems 7  
2.2. Root systems of algebraic groups 9  
2.3. Root subgroups 10  
2.4. Parabolic subgroups 11  
2.5. The Weyl group of a parabolic subgroup 12  
2.6. Schubert varieties 12  
2.7. The Bruhat–Chevalley order 13  
2.8. Line bundles on $G/Q$ 13  
2.9. Geometric properties of Schubert varieties 15  
2.10. Equations defining a Schubert variety 16  
2.11. Representations of semisimple algebraic groups 17  

Chapter 3. Specifics for the Classical Groups 23  
3.1. The Grassmannian variety $G_{d,n}$ 23  
3.2. The special linear group $SL(n)$ 26  
3.3. The symplectic group $Sp(2n)$ 29  
3.4. The odd orthogonal group $SO(2n + 1)$ 31  
3.5. The even orthogonal group $SO(2n)$ 33  

Chapter 4. The Tangent Space and Smoothness 37  
4.1. The Zariski tangent space 37  
4.2. Smooth and singular points 37  
4.3. The space $T(w,\tau)$ 38  
4.4. A canonical affine neighborhood of a $T$-fixed point 39  
4.5. Tangent cone and Jacobian criteria for smoothness 40  
4.6. Discussion of smoothness at a $T$-fixed point 41  
4.7. Multiplicity at a point $P$ on a variety $X$ 42  
4.8. Degree of $X(w)$ 44  
4.9. Summary of smoothness criteria 46  

Chapter 5. Root System Description of $T(w,\tau)$ 47  
5.1. Polo's results 47
5.2. Bases $B_\lambda$, $B_\lambda^*$ for $V_K(\lambda)$ and $H^0(G/B, L_\lambda)$ 49
5.3. Description of $T(w, \text{id})$ 52
5.4. Description of $T(w, \tau)$ 56
5.5. Tangent space and certain weight multiplicities 63
5.6. The $B$-module $T(w, \text{id})$ 67
5.7. Two smoothness criteria of Carrell–Kuttler 68

Chapter 6. Rational Smoothness and Kazhdan–Lusztig Theory 71
6.1. Kazhdan–Lusztig polynomials 72
6.2. Carrell–Peterson's criteria 77
6.3. Combinatorial formulas for Kazhdan–Lusztig polynomials 81

Chapter 7. Nil-Hecke Ring and the Singular Locus of $X(w)$ 91
7.1. The nil-Hecke ring 91
7.2. Criteria for smoothness and rational smoothness 94
7.3. Representation-theoretic results on the tangent cone 97
7.4. Proof of smoothness criterion 98

Chapter 8. Patterns, Smoothness and Rational Smoothness 103
8.1. Type $A$: criterion in terms of patterns 103
8.2. Conjecture in type $A$ 104
8.3. Types $B$, $C$, $D$: criterion in terms of patterns 106
8.4. Type $C$ results of Lakshmibai–Song using permutations 115

Chapter 9. Minuscule and cominuscule $G/P$ 119
9.1. Results on small resolutions 122
9.2. Brion–Polo results 131
9.3. Irreducible components of $\text{Sing}X(w)$ in special cases 138
9.4. Multiplicity at a singular point 144
9.5. The symplectic Grassmannian $Sp(2n)/P_n$ 155

Chapter 10. Rank Two Results 159
10.1. Kumar's method 159
10.2. Tangent space computations 161

Chapter 11. Related Combinatorial Results 169
11.1. Factoring the Poincaré polynomial of a Schubert variety 169
11.2. Structure of Bruhat intervals 170
11.3. Generating function for smooth permutations 172
11.4. Bona's results 172

Chapter 12. Related Varieties 175
12.1. Opposite cells in Schubert varieties in $SL(n)/B$ 175
12.2. Determinantal varieties 180
12.3. Ladder determinantal varieties 185
12.4. Quiver varieties 201