# Contents

## Preface

1 Introduction 1

1.1 KKM Theory: An Outline 1

1.2 Generic Study Method for Nonlinear Problems 18

1.3 Characterization of Set-Valued Mappings and Strong Separation Property 21

1.3.1 Characterization of Upper Hemicontinuous Set-valued Mappings 22

1.3.2 Characterization of Strong Separation Property 28

2 The KKM Theory and Related Topics 37

2.1 Introduction 38

2.2 Knaster-Kuratowski-Mazurkiewicz Principle 41

2.3 Ky Fan Minimax Principle 51

2.4 Browder-Fan Fixed Points in Product Spaces 62

2.4.1 Fixed Points of Couple-Majorized Mappings 62

2.4.2 Browder-Fan Fixed Points of $L_s$-Majorized Mappings 71

2.5 Generic Stability of Ky Fan Points 84

2.5.1 Stability Results in Compact Setting 87

2.5.2 Stability Results in Non-Compact Setting 89

2.5.3 Essential Components of $KF$ Points 94

2.6 Coincident Points in Topological Vector Spaces 96

2.6.1 Some Facts 97

2.6.2 Fan's Best Approximation Theorems 101

2.6.3 Coincidence Theorems 107

2.7 Matching Theorems and Applications 113

2.7.1 Fan-Glicksberg Fixed Points in Topological Vector Spaces 113

2.7.2 Matching Theorems for Closed Coverings of Convex Sets 124

2.8 Generic Stability of Coincident Points of Set-Valued Mappings 127

2.8.1 Generic Stability of Coincident Points 127

2.8.2 Essential Components of Coincident Points 132

2.9 Ky Fan Section Theorems 140

2.10 Fan's Best Approximation and Fan-Glicksberg Fixed Points in Locally Convex $H$-Spaces 149
## Contents

### 2.10.1 Fan’s Best Approximation and Fixed Points in Locally Convex Spaces
- 2.10.2 Fan-Glicksberg Fixed Points in Locally Convex $H$-Spaces

### 2.11 KKM Theory in Hyperconvex Metric Spaces and Some Applications
- 2.11.1 Hyperconvex Spaces and Fan-Glicksberg Fixed Points
- 2.11.2 KKM Principle
- 2.11.3 Intersection Theorems
- 2.11.4 Fixed Point Theorems
- 2.11.5 Saddle Points and Nash Equilibria

### 2.12 Fixed Point Theorems of Set-Valued Mappings in Abstract Convex Spaces
- 2.12.1 MC-Space: An Abstract Structure
- 2.12.2 Browder-Fan Fixed Points in MC-Spaces
- 2.12.3 Fan-Glicksberg Fixed Points in MC-Spaces
- 2.12.4 Fan-Glicksberg Fixed Points in G-Convex Spaces
- 2.12.5 Generic Stability of Fixed Points in Hyperconvex Metric Spaces

### 3 Topological Intersection and Minimax Inequality Theory
- 3.1 Introduction
- 3.2 Characterization of Topological Intersection Theorems
- 3.3 Topological Intersection of Set-Valued Mappings without Closed Graphs
- 3.4 Topological Intersection of Two Set-Valued Mappings
- 3.5 Topological Minimax Inequalities

### 4 Abstract Economic Models and Generalized Games
- 4.1 Introduction
- 4.2 Abstract Economics in $H$-Spaces
- 4.3 The Study of Abstract Economics via KKM Theory
- 4.3.1 Introduction
- 4.3.2 Browder-Fan Fixed Point Theorems
- 4.3.3 Existence of Maximal Elements for $F$-majorized Mappings
- 4.3.4 Equilibria in Topological Vector Spaces
- 4.3.5 The Concept of Approximate Equilibria
- 4.4 Approximate Equilibria in Topological Vector Spaces
- 4.5 Abstract Economics in Locally Convex Spaces
- 4.5.1 Equilibria of Abstract Economics for $KF$-Majorized Mappings in Locally Convex Spaces
- 4.5.2 Equilibria of Abstract Economics for $L$-Majorized Mappings in Locally Convex Spaces
## Contents

4.6 Abstract Economics without Compactness and Paracompactness 288  
4.7 Equilibria of \( \mathcal{U} \)-Majorized Mappings 300  
4.8 Equilibria of \( \Psi \)-Condensing Mappings 306  
4.9 Abstract Economics in Frechet Spaces 312  
4.10 Abstract Economics 315  
  4.10.1 Properties of Lower Semicontinuous Set-Valued Mappings 316  
  4.10.2 Equilibria 321  

5 Variational Inequalities - Part I 325  
  5.1 Introduction 326  
  5.2 Variational Inequalities in Locally Convex Spaces 329  
  5.3 Generalized Quasi-Variational Inequalities 334  
  5.4 Stability of Quasi-Variational Inequalities 343  
    5.4.1 Generic Stability of Quasi-Variational Inequalities 343  
    5.4.2 Essentially Connected Components 351  
  5.5 Generalized Vector-Valued Variational-Like Inequalities 356  
  5.6 Algorithms of Generalized Quasi-Variational Inclusion 366  

6 Variational Inequalities - Part II 381  
  6.1 Introduction 381  
  6.2 Non-Compact Generalized Quasi-Variational Inequalities 382  
  6.3 Generalized Quasi-Variational Inequalities without Upper Semicontinuity 390  
  6.4 Non-Compact Variational Inequalities in Locally Convex Spaces 395  
  6.5 General Quasi-Variational Inequalities and Applications 400  
  6.6 Generalized Quasi-Variational Inequalities of Monotone and Quasi-Monotone Mappings 411  
  6.7 Variational Inequalities in Banach Spaces 419  
  6.8 Variational Inequalities of Quasi-Monotone Mappings in Banach Spaces 430  
  6.9 Applications to Fixed Points and Minimization Problems 433  
  6.10 Implicitly Quasi-Variational Inequalities 439  
  6.11 Applications to Complementarity Problems 445  
  6.12 Stability Results of Nonlinear Complementarity Problems 451  
    6.12.1 Generic Stability of Nonlinear Complementarity Problems 451  
    6.12.2 Existence of Essentially Connected Components 456  

7 Applications to Multi-Objective Optimization 463  
  7.1 Introduction 463  
  7.2 \( \varepsilon \)-Saddle Points in Topological Vector Spaces 464  
  7.3 Saddle Points Beyond Topological Vector Spaces 470  
  7.4 Vector-Valued Saddle Points in Topological Vector Spaces 475  
  7.5 Vector-Valued Saddle Points in \( H \)-Spaces 486
7.6 Generic Stability of Vector-Valued Saddle Points ........................................ 499
  7.6.1 Stability Results in Compact Case ....................................................... 500
  7.6.2 Stability Results in Non-Compact Case ................................................ 503
7.7 Existence of Vector-Valued Minimax Inequalities ........................................ 508

8 Applications to Game Theory ................................................................. 519
  8.1 Introduction ............................................................................................. 519
  8.2 Weight Nash Equilibria of Multi-Objective Games ...................................... 520
    8.2.1 Fixed Point Approach ........................................................................ 522
    8.2.2 Ky Fan Minimax Inequality Method ................................................... 525
  8.3 Nash-Equilibria of Non-Cooperative $N$-Person Games ............................... 530
  8.4 Generic Stability of Nash Equilibria ......................................................... 535

9 Applications to Mathematical Economics ..................................................... 545
  9.1 Introduction ............................................................................................. 545
  9.2 Debreu-Gale-Nikaido Lemma ..................................................................... 546
    9.2.1 Compact Case ..................................................................................... 547
    9.2.2 Non-Compact Case ............................................................................ 551
  9.3 Existence of Pareto Optima ...................................................................... 553
    9.3.1 Set-Valued Dynamic Systems ............................................................. 554
    9.3.2 Applications to the Existence of Pareto Optima .................................. 559
  9.4 Stability of Pareto Optima ........................................................................ 564
  9.5 Generic Stability of Production Economies ................................................. 566

References .................................................................................................. 575

Index .......................................................................................................... 613