Vortices in Nonlinear Fields

From Liquid Crystals to Superfluids
From Non-Equilibrium Patterns to Cosmic Strings

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