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The Dynamical System Generated by the 3n+1 Function

Springer
THE DYNAMICAL SYSTEM
ON THE NATURAL NUMBERS
GENERATED BY THE $3n + 1$ FUNCTION

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