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with Appendices by M. Katz, P. Pansu, and S. Semmes

Metric Structures for Riemannian and Non-Riemannian Spaces

Based on Structures Métriques des Variétés Riemanniennes
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