Generalized Vector and Dyadic Analysis
Applied Mathematics in Field Theory
Second Edition

Chen-To Tai
Professor Emeritus
Radiation Laboratory
Department of Electrical Engineering
and Computer Science
University of Michigan

IEEE Antennas & Propagation Society, Sponsor

IEEE PRESS
The Institute of Electrical and Electronics Engineers, Inc.
New York

Oxford University Press
Oxford, Tokyo, Melbourne
## Contents

Preface to the Second Edition xi  
Preface to the First Edition xiii  
Acknowledgments for the First Edition xv  

### 1 Vector and Dyadic Algebra 1

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-1</td>
<td>Representations of Vector Functions</td>
<td>1</td>
</tr>
<tr>
<td>1-2</td>
<td>Products and Identities</td>
<td>4</td>
</tr>
<tr>
<td>1-3</td>
<td>Orthogonal Transformation of Vector Functions</td>
<td>8</td>
</tr>
<tr>
<td>1-4</td>
<td>Transform of Vector Products</td>
<td>14</td>
</tr>
<tr>
<td>1-5</td>
<td>Definition of Dyadics and Tensors</td>
<td>16</td>
</tr>
<tr>
<td>1-6</td>
<td>Classification of Dyadics</td>
<td>17</td>
</tr>
<tr>
<td>1-7</td>
<td>Products Between Vectors and Dyadics</td>
<td>19</td>
</tr>
</tbody>
</table>

### 2 Coordinate Systems 23

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-1</td>
<td>General Curvilinear System (GCS)</td>
<td>23</td>
</tr>
<tr>
<td>2-2</td>
<td>Orthogonal Curvilinear System (OCS)</td>
<td>28</td>
</tr>
</tbody>
</table>
3  Line Integrals, Surface Integrals, and Volume Integrals  43

3-1  Differential Length, Area, and Volume  43
3-2  Classification of Line Integrals  44
3-3  Classification of Surface Integrals  48
3-4  Classification of Volume Integrals  56

4  Vector Analysis in Space  58

4-1  Symbolic Vector And Symbolic Vector Expressions  58
4-2  Differential Formulas of the Symbolic Expression in the Orthogonal Curvilinear Coordinate System for Gradient, Divergence, and Curl  61
4-3  Invariance of the Differential Operators  65
4-4  Differential Formulas of the Symbolic Expression in the General Curvilinear System  69
4-5  Alternative Definitions of Gradient and Curl  75
4-6  The Method of Gradient  78
4-7  Symbolic Expressions with Two Functions and the Partial Symbolic Vectors  81
4-8  Symbolic Expressions with Double Symbolic Vectors  86
4-9  Generalized Gauss Theorem in Space  91
4-10  Scalar and Vector Green's Theorems  93
4-11  Solenoidal Vector, Irrotational Vector, and Potential Functions  95

5  Vector Analysis on Surface  99

5-1  Surface Symbolic Vector and Symbolic Expression for a Surface  99
5-2  Surface Gradient, Surface Divergence, and Surface Curl  101
5-2-1  Surface Gradient  101
5-2-2 Surface Divergence 102
5-2-3 Surface Curl 103
5-3 Relationship Between the Volume and Surface Symbolic Expressions 104
5-4 Relationship Between Weatherburn's Surface Functions and the Functions Defined in the Method of Symbolic Vector 104
5-5 Generalized Gauss Theorem for a Surface 106
5-6 Surface Symbolic Expressions with a Single Symbolic Vector and Two Functions 111
5-7 Surface Symbolic Expressions with Two Surface Symbolic Vectors and a Single Function 113

6 Vector Analysis of Transport Theorems 116

6-1 Helmholtz Transport Theorem 116
6-2 Maxwell Theorem and Reynolds Transport Theorem 119

7 Dyadic Analysis 121

7-1 Divergence and Curl of Dyadic Functions and Gradient of Vector Functions 121
7-2 Dyadic Integral Theorems 124

8 A Historical Study of Vector Analysis 127

8-1 Introduction 127
8-2 Notations and Operators 129
8-2-1 Past and Present Notations in Vector Analysis 129
8-2-2 Quaternion Analysis 131
8-2-3 Operators 132
8-3 The Pioneer Works of J. Willard Gibbs (1839–1903) 135
8-3-1 Two Pamphlets Printed in 1881 and 1884 135
8-3-2 Divergence and Curl Operators and Their New Notations 138
8-4 Book by Edwin Bidwell Wilson Founded Upon the Lectures of J. Willard Gibbs 141
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>8-4-1</td>
<td>Gibbs's Lecture Notes</td>
<td>141</td>
</tr>
<tr>
<td>8-4-2</td>
<td>Wilson's Book</td>
<td>141</td>
</tr>
<tr>
<td>8-4-3</td>
<td>The Spread of the Formal Scalar Product (FSP) and Formal Vector Product (FVP)</td>
<td>146</td>
</tr>
<tr>
<td>8-5</td>
<td>( \nabla ) in the Hands of Oliver Heaviside (1850–1925)</td>
<td>149</td>
</tr>
<tr>
<td>8-6</td>
<td>Shilov's Formulation of Vector Analysis</td>
<td>151</td>
</tr>
<tr>
<td>8-7</td>
<td>Formulations in Orthogonal Curvilinear Systems</td>
<td>152</td>
</tr>
<tr>
<td>8-7-1</td>
<td>Two Examples from the Book by Moon and Spencer</td>
<td>152</td>
</tr>
<tr>
<td>8-7-2</td>
<td>A Search for the Divergence Operator in Orthogonal Curvilinear Coordinate Systems</td>
<td>154</td>
</tr>
<tr>
<td>8-8</td>
<td>The Use of ( \nabla ) to Derive Vector Identities</td>
<td>155</td>
</tr>
<tr>
<td>8-9</td>
<td>A Recasting of the Past Failures by the Method of Symbolic Vector</td>
<td>157</td>
</tr>
<tr>
<td>8-9-1</td>
<td>In Retrospect</td>
<td>159</td>
</tr>
</tbody>
</table>

### Appendix A

Transformation Between Unit Vectors 161

### Appendix B

Vector and Dyadic Identities 165

### Appendix C

Integral Theorems 169

### Appendix D

Relationships Between Integral Theorems 170

### Appendix E

Vector Analysis in the Special Theory of Relativity 174

### Appendix F

Comparison of the Nomenclatures and Notations of the Quantities Used in This Book and in the Book by Stratton 181

References 185

Index 189