Peter Bürgisser Michael Clausen M. Amin Shokrollahi

Algebraic Complexity Theory

With the Collaboration of Thomas Lickteig

With 21 Figures



Table of Contents

Chapu	1. Introduction	1
1.1	Exercises	20
1.2	Open Problems	23
1.3	Notes	23
Part I	. Fundamental Algorithms	
Chapte	er 2. Efficient Polynomial Arithmetic	27
2.1	Multiplication of Polynomials I	28
2.2*	Multiplication of Polynomials II	34
2.3*	Multiplication of Several Polynomials	38
2.4	Multiplication and Inversion of Power Series	44
2.5*	Composition of Power Series	47
2.6	Exercises	53
2.7	Open Problems	57
2.8	Notes	58
Chapt	er 3. Efficient Algorithms with Branching	61
3.1	Polynomial Greatest Common Divisors	61
3.2*	Local Analysis of the Knuth-Schönhage Algorithm	71
3.3	Evaluation and Interpolation	75
3.4*	Fast Point Location in Arrangements of Hyperplanes	79
3.5*	Vapnik-Chervonenkis Dimension and Epsilon-Nets	84
3.6	Exercises	90
3.7	Open Problems	97
3.8	Notes	98
Part :	II. Elementary Lower Bounds	
Chapt	ter 4. Models of Computation	103
4.1	Straight-Line Programs and Complexity	103
4.2	Computation Sequences	
4.3*	Autarky	

XX	Table of Contents	
XX	lable of Contents	

9	4.4* 4.5* 4.6 4.7	Computation Trees	118 121
	Chapte 5.1 5.2 5.3* 5.4 5.5 5.6	Preconditioning and Transcendence Degree Preconditioning. Transcendence Degree Extension to Linearly Disjoint Fields Exercises Open Problems Notes	125 130 134 136 142
	Chapte 6.1 6.2 6.3* 6.4 6.5 6.6	Discussion of Ideas Lower Bounds by the Degree of Linearization Continued Fractions, Quotients, and Composition Exercises Open Problems Notes	144 148 151 157 159
	7.1 7.2 7.3 7.4 7.5	Complexity of Truncated Taylor Series Complexity of Partial Derivatives Exercises Open Problems Notes II. High Degree	161 164 167 168
	Chapte 8.1 8.2 8.3 8.4 8.5* 8.6* 8.7 8.8 8.9	er 8. The Degree Bound	171 178 182 186 192 195 198 205 205 207 207 211

	Table of Contents	XXI
9.4* 9.5* 9.6 9.7 9.8	Polynomials with Rapidly Growing Integer Coefficients Extension to other Complexity Measures Exercises Open Problems Notes	236 243
Chapte	er 10. Branching and Degree	245
10.1	Computation Trees and the Degree Bound	
10.2	Complexity of the Euclidean Representation	
10.3*	Degree Pattern of the Euclidean Representation	
10.4	Exercises	
10.5	Open Problems	
10.6	Notes	264
Chapte	er 11. Branching and Connectivity	265
11.1*	Estimation of the Number of Connected Components	265
11.2	Lower Bounds by the Number of Connected Components	272
11.3	Knapsack and Applications to Computational Geometry	
11.4	Exercises	
11.5	Open Problems	
11.6	Notes	283
Chapte	er 12. Additive Complexity	287
12.1	Introduction	
12.2*	Real Roots of Sparse Systems of Equations	289
12.3	A Bound on the Additive Complexity	
12.4	Exercises	
12.5	Open Problems	
12.6	Notes	301
Part I	V. Low Degree	
Chapte	er 13. Linear Complexity	305
13.1	The Linear Computational Model	
13.2	First Upper and Lower Bounds	
13.3*	A Graph Theoretical Approach	
13.4*	Lower Bounds via Graph Theoretical Methods	
13.5*	Generalized Fourier Transforms	
13.6	Exercises	
13.7 13.8	Open Problems	
•	er 14. Multiplicative and Bilinear Complexity	
14.1	Multiplicative Complexity of Quadratic Maps	
14.2	The Tensorial Notation	357

.

XXII Table of Contents

14.3	Restriction and Conciseness	361
14.4	Other Characterizations of Rank	365
14.5	Rank of the Polynomial Multiplication	367
14.6*	The Semiring T	368
14.7	Exercises	
14.8	Open Problems	373
14.9	Notes	
Chapte	er 15. Asymptotic Complexity of Matrix Multiplication	
15.1	The Exponent of Matrix Multiplication	
15.2	First Estimates of the Exponent	
15.3	Scalar Restriction and Extension	381
15.4	Degeneration and Border Rank	384
15.5	The Asymptotic Sum Inequality	389
15.6	First Steps Towards the Laser Method	391
15.7*	Tight Sets	
15.8	The Laser Method	
15.9*	Partial Matrix Multiplication	
	Rapid Multiplication of Rectangular Matrices	
15.11	Exercises	
15.12	Open Problems	
15.13	Notes	
13.13	11000	120
Chapte	er 16. Problems Related to Matrix Multiplication	425
16.1	Exponent of Problems	425
16.2	Triangular Inversion	427
16.3	LUP-decomposition	428
16.4	Matrix Inversion and Determinant	430
16.5*	Transformation to Echelon Form	431
16.6*	The Characteristic Polynomial	
16.7*	Computing a Basis for the Kernel	
16.8*	Orthogonal Basis Transform	
16.9*	Matrix Multiplication and Graph Theory	
16.10	Exercises	
16.11	Open Problems	
16.12	Notes	
Chapte	er 17. Lower Bounds for the Complexity of Algebras	455
17.1	First Steps Towards Lower Bounds	455
17.2	Multiplicative Complexity of Associative Algebras	463
17.3*	Multiplicative Complexity of Division Algebras	470
17.4*	Commutative Algebras of Minimal Rank	474
17.5	Exercises	481
17.6	Open Problems	484
17.7	Notes	

F	er 18. Rank over Finite Fields and Codes	489
18.1	Linear Block Codes	489
18.2		491
18.3	Polynomial Multiplication over Finite Fields	492
18.4*		494
18.5*		496
18.6		498
18.7	Open Problems	502
18.8	Notes	
-	er 19. Rank of 2-Slice and 3-Slice Tensors	
19.1	The Weierstraß-Kronecker Theory	
19.2	Rank of 2-Slice Tensors	
19.3*	Rank of 3-Slice Tensors	
19.4	Exercises	
19.5	Notes	519
Chapte	er 20. Typical Tensorial Rank	521
20.1	Geometric Description	
20.2	Upper Bounds on the Typical Rank	
20.3*	Dimension of Configurations in Formats	
20.4	Exercises	
20.5	Open Problems	
20.6*	Appendix: Topological Degeneration	
20.7	Notes	534
20.7	Notes	539
	· · · · · · · · · · · · · · · · · · ·	539
		539
Part V	V. Complete Problems	
Part V	V. Complete Problems er 21. P Versus NP: A Nonuniform Algebraic Analogue	543
Part V Chapte 21.1	V. Complete Problems er 21. P Versus NP: A Nonuniform Algebraic Analogue Cook's Versus Valiant's Hypothesis	543 543
Part V Chapte 21.1 21.2	V. Complete Problems er 21. P Versus NP: A Nonuniform Algebraic Analogue Cook's Versus Valiant's Hypothesis p-Definability and Expression Size	543 543 550
Part V Chapte 21.1 21.2 21.3	V. Complete Problems er 21. P Versus NP: A Nonuniform Algebraic Analogue Cook's Versus Valiant's Hypothesis p-Definability and Expression Size Universality of the Determinant	543 543 550 554
Part V Chapte 21.1 21.2 21.3 21.4	V. Complete Problems er 21. P Versus NP: A Nonuniform Algebraic Analogue Cook's Versus Valiant's Hypothesis p-Definability and Expression Size Universality of the Determinant Completeness of the Permanent	543 543 550 554 556
Part V Chapte 21.1 21.2 21.3 21.4 21.5*	er 21. P Versus NP: A Nonuniform Algebraic Analogue Cook's Versus Valiant's Hypothesis p-Definability and Expression Size Universality of the Determinant Completeness of the Permanent The Extended Valiant Hypothesis	543 543 550 554 556 561
Chapte 21.1 21.2 21.3 21.4 21.5* 21.6	er 21. P Versus NP: A Nonuniform Algebraic Analogue Cook's Versus Valiant's Hypothesis p-Definability and Expression Size Universality of the Determinant Completeness of the Permanent The Extended Valiant Hypothesis Exercises	543 543 550 554 556 561 569
Chapte 21.1 21.2 21.3 21.4 21.5* 21.6 21.7	er 21. P Versus NP: A Nonuniform Algebraic Analogue Cook's Versus Valiant's Hypothesis p-Definability and Expression Size Universality of the Determinant Completeness of the Permanent The Extended Valiant Hypothesis Exercises Open Problems	543 543 550 554 560 560 574
Chapte 21.1 21.2 21.3 21.4 21.5* 21.6	er 21. P Versus NP: A Nonuniform Algebraic Analogue Cook's Versus Valiant's Hypothesis p-Definability and Expression Size Universality of the Determinant Completeness of the Permanent The Extended Valiant Hypothesis Exercises	543 543 550 554 560 560 574
Chapte 21.1 21.2 21.3 21.4 21.5* 21.6 21.7 21.8	er 21. P Versus NP: A Nonuniform Algebraic Analogue Cook's Versus Valiant's Hypothesis p-Definability and Expression Size Universality of the Determinant Completeness of the Permanent The Extended Valiant Hypothesis Exercises Open Problems	543 543 550 554 556 561 569 574
Chapte 21.1 21.2 21.3 21.4 21.5* 21.6 21.7 21.8	er 21. P Versus NP: A Nonuniform Algebraic Analogue Cook's Versus Valiant's Hypothesis p-Definability and Expression Size Universality of the Determinant Completeness of the Permanent The Extended Valiant Hypothesis Exercises Open Problems Notes	543 550 554 556 569 574 577
Chapte 21.1 21.2 21.3 21.4 21.5* 21.6 21.7 21.8 Biblio	er 21. P Versus NP: A Nonuniform Algebraic Analogue Cook's Versus Valiant's Hypothesis p-Definability and Expression Size Universality of the Determinant Completeness of the Permanent The Extended Valiant Hypothesis Exercises Open Problems Notes	543 550 554 556 561 574 577 601