# Absolutely Summing <br> Operators 

Joe Diestel<br>Department of Mathematics and Computer Science<br>Kent State University

Hans Jarchow<br>Mathematisches Institut<br>Universität Zürich

Andrew Tonge<br>Department of Mathematics and Computer Science Kent State University

## Contents

Introduction ..... xi
Notation ..... xiii

1. Unconditional and Absolute Summability in Banach Spaces ..... 1
The Dvoretzky-Rogers Theorem ..... 1
Absolutely convergent series are unconditionally convergent in Banach spaces, Dvoretzky - Rogers Theorem, Coincidence of absolute and unconditional summa- bility only in finite dimensional spaces
Unconditional Convergence and the Orlicz- Pettis Theorem ..... 4
Simple characterizations of unconditional summability, Bounded Multiplier Test, Schur's $\ell_{1}$ Theorem, Orlicz - Pettis Theorem, Omnibus Theorem on unconditional summability
Khinchin's Inequality ..... 9
Rademacher functions, Khinchin's Inequality, Orlicz's Theorem on unconditional
Rademacher functions, Khinchin's Inequality, Orlicz's Theorem on unconditional summability in $L_{1}$, Span of the Rademacher functions in $L_{p}[0,1]$ summability in $L_{1}$, Span of the Rademacher functions in $L_{p}[0,1]$
Grothendieck's Inequality ..... 15
Absolutely summing operators, Grothendieck's Theorem on operators from $\ell_{1}$ to $\ell_{2}$, Grothendieck's Inequality, Grothendieck's Theorem on finite dimensional spaces Notes and Remarks ..... 19
2. Fundamentals of $\boldsymbol{p}$-Summing Operators ..... 31
Definition ..... 31
$p$-summing operators, $p$-summing norm
Vector-Valued Sequence Spaces ..... 32
Strong $\ell_{p}$ sequences, Weak $\ell_{p}$ sequences, Characterization of $p$-summing operators Constructions of p-Summing Operators ..... 36
Finite rank operators, The Banach ideal of $p$-summing operators, Injectivity, Inclusion Theorem
Basic Examples ..... 40
Multiplication operators, Formal inclusion operators, Diagonal operators, Embeddings of function spaces, Kernel operators
Domination and Factorization ..... 43
Pietsch Domination Theorem, Pietsch Factorization Theorem, Operators from and to $\mathcal{C}(K)$-spaces, 2 -summing operators
Some Consequences49
Weak compactness and complete continuity of $p$-summing operators, Weak Dvoretzky-Rogers Theorem, $p$-summing character of biadjoints and adjoints
Composition ..... 52
Notes and Remarks ..... 55
3. Summing Operators on $\mathcal{L}_{p}$-Spaces ..... 60
$\mathcal{L}_{p}$-Spaces ..... 60Operators from $\mathcal{L}_{1}$-spaces to $\mathcal{L}_{2}$-spaces are 1-summing, Approximation in $L_{p}(\mu)$and $\mathcal{C}(K), L_{p}(\mu)$ and $\mathcal{C}(K)$ as basic examples of $\mathcal{L}_{p}$-spaces
Operators on $\mathcal{L}_{\infty}$-Spaces64
Operators from $\mathcal{L}_{\infty}$-spaces to $\mathcal{L}_{p}$-spaces ( $1 \leq p \leq 2$ ) are 2 -summing Some Applications ..... 66
Quotients of $\mathcal{C}(K)$ which are subspaces of $\mathcal{L}_{1}$, Coincidence of 2 -summing and 1 -sum-ming operators on subspaces of $\mathcal{L}_{p}(1 \leq p \leq 2)$, uniqueness of unconditional basis in $\ell_{1}$,
$L_{1}[0,1]$ has no unconditional basis, Coincidence of $q$-summing and 2-summing operators on subspaces of $\mathcal{L}_{p}$ for $1 \leq p \leq 2<q<\infty$, Extrapolation Theorem Notes and Remarks ..... 73
4. Operators on Hilbert Spaces and Summing Operators ..... 76
Compact Hilbert Space Operators ..... 76
Spectral Theorem, Orthonormal Representation, Approximation Numbers
Schatten - von Neumann classes ..... 80General Theory, Hilbert - Schmidt operators
Hilbert-Schmidt Operators and Summing Operators ..... 84Coincidence with 2 -summing operators, with $p$-summing operators,Characterization by factorization
Extension Property ..... 85
Injective Banach spaces, $\Pi_{2}$-Extension Theorem, Kadets-Snobar Theorem
ADJOINTS OF 2-SUMMING OPERATORS ..... 88
Notes and Remarks ..... 90
5. $p$-Integral Operators ..... 95
Definition and Elementary Properties ..... 95
p-integral operators, Banach ideal property, strictly p-integral operators
Relations to $\boldsymbol{p}$-Summing Operators ..... 97
$p$-integral operators are $p$-summing, Conditions for the converse, Further cases of coincidence$p$-Summing Operators Failing to be p-Integral99
Further Structural Results ..... 104$p$-integrality of second adjoints, The case $p=1$, Composition of summing andintegral operators
ORDER BoUndedness ..... 107
Definitions, $L_{p}(\mu)$-valued order bounded operators are $p$-integral, Converse for $p=1$, Operators with $p$-summing adjoint$p$-Nuclear Operators111
Various characterizations of $p$-nuclear operators, Banach ideal property
Relations to Integral Operators ..... 114
p-nuclear operators are p-integral, Cases of coincidence, Factorization, Composition of summing and nuclear operators
Hilbert Space Operators ..... 118
$p$-nuclear and $p$-integral operators on Hilbert space, Composition of 2-summing operators
Notes and REMARKs ..... 119
6. Trace Duality ..... 125
Trace ..... 125
Trace and trace duality in finite dimensions
Duality for Schatten - von Neumann Classes ..... 126Composition Theorem, Trace and duality for Schatten - von Neumann classesBanach Ideals130Operator ideals and their norms, Banach ideals, Comparison
ADJOINT IDEALS ..... 132Adjoint ideals, Biadjoint criterionMaximal Ideals134Ordering Banach ideals, Maximality, Maximality and adjoints, Maximality of$p$-integral operators, Adjoints of $p$-integral operators, Maximality of $q$-summingoperators, Minimality of $p$-nuclear operators, Characterization of $\mathcal{L}_{\infty}$-spaces
Applications ..... 143
Lewis' Theorem, Auerbach's Lemma, John's ellipsoids, Carathéodory's Theorem Notes and Remarks ..... 151
7. 2-Factorable Operators ..... 154
Generalities ..... 154
The Banach ideal of $p$-factorable operators, adjoints, the case $p=2$
Maximality of $\left[\Gamma_{2}, \gamma_{2}\right]$ ..... 156
Relations with Grothendieck's Inequality ..... 157
Matricial characterization, Preparations for Maurey's Extension Theorem
2-Dominated Operators ..... 162
Definition, Maximality, Trace duality of 2-dominated and 2-factorable operators,Further characterizations
Notes and Remarks ..... 166
8. Ultraproducts and Local Reflexivity ..... 169
Generalities on Ultraproducts ..... 169
Ultraproduct of Banach spaces and operators, Ultrapowers, Ultraproducts of finite dimensional spaces
Some Stability Properties ..... 172
Banach lattices, $L_{p}(\mu)$-spaces, $\mathcal{C}(K)$-spaces, Banach algebras and ultraproducts
Ultraproducts and Finite Dimensional Structure ..... 173
Banach spaces as subspaces of ultraproducts of finite dimensional subspaces, Representation of operators by ultraproducts of finite dimensional operators Finite Representability ..... 175
Concepts, Relation with ultraproducts, Application to $L_{p}(\mu)$ - and $\mathcal{C}(K)$-spaces
Local Reflexivity177
Helly's Lemma, Principle of Local Reflexivity, Application to biduals Notes and Remarks ..... 182
9. $p$-Factorable Operators ..... 185
Maximality of $\Gamma_{p}$ ..... 185
Maximality, Inclusions
Dual IDEALS ..... 186
Definition, Adjoints and maximality of dual ideals
$q$-DOMINATED OPERATORS ..... 187
Definition and characterizations, Maximality, Trace duality of $p$-factorable and $p^{*}$-dominated operators
Proof of the Main Result ..... 190
Ky Fan's Lemma
Applications ..... 192
Characterization of complemented subspaces, subspaces and quotients of $L_{p}(\mu)$-spaces, Extensions of $p$-factorable operators
Notes and Remarks ..... 195
10. ( $q, p$ )-Summing Operators ..... 197
Some Basic Properties ..... 197
Fundamentals, Banach ideal property, ( $q, 2$ )-summing operators and Schatten- von Neumann classes, Inclusion theorem, Dvoretzky - Rogers Theorem OPERATORS ON $\mathcal{L}_{\infty}$-SPACES ..... 199
Operators from $\mathcal{L}_{\infty}$ to $\mathcal{L}_{p}, p>2$, Results of interpolation and factorization for( $q, p$ )-summing operators on $\mathcal{C}(K)$-spaces, Coincidence of ( $q, 1$ )-summing and( $q, p$ )-summing operators on $\mathcal{C}(K)$-spaces ( $q>p$ ), Applications
Notes and Remarks207
11. Type and Cotype: The Basics ..... 211
Kahane's InequalityRandomized Sums212Fundamentals, Lévy's Inequality
Rademacher Sums ..... 214
Rademacher Sums in $\ell_{r}$ ..... 216
Proof of Kahane's Inequality
Type and Cotype ..... 217
Definitions, Type and Cotype of $\mathcal{L}_{r}$-spaces, Permanence Properties, Type and cotype of Lebesgue-Bochner spaces
Summing Operators ..... 222
Cotype and inclusion theorems for summing operators, Cotype and summing operators on $\mathcal{C}(K)$-spaces, Orlicz's Theorem and cotype, Subspaces of $L_{1}$ having type >1
Notes and Remarks ..... 225
12. Randomized Series and Almost Summing Operators ..... 230
Randomized Series ..... 230
Almost sure summability, Standard characterizations
Rademacher Series ..... 231Contraction Principle, Almost sure summability and convergence in $L_{p}(X)$, TheBanach space $\operatorname{Rad}(X)$, Relations to type and cotype
Almost Summing Operators ..... 234
Definition, $p$-summing operators are almost summing, Almost summing operators and cotype, Ideal properties of almost summing operators, Type 2 and almost summing operators
Gaussian Variables ..... 237
Applications to Almost Summing Operators ..... 239
Characterization of almost summing operators using Gaussian variables, The $\gamma$-summing norm, 2-dominated and almost summing operators
Some Consequences ..... 246
Kwapień's Theorem, Maurey's Extension Theorem, Applications248
Definitions, Relations to type and cotype
The Maurey - Rosenthal Theorem ..... 251
Statement of the theorem, A dilation theorem, Bennett-Maurey - Nahoum decom- position of unconditionally summable sequences in $L_{1}$255
13. K-Convexity and B-Convexity ..... 258
K-Convexity ..... 259Definitions, Uniform containment of $\ell_{1}^{n}$ 's
B-convexity ..... 261Fundamentals, Characterization by uniform containment of $\ell_{1}^{n}$ 's, B-convexityand duality, B-convexity and type
Equivalence of B- and K-Convexity ..... 267
Semigroups of operators, Beurling - Kato Theorem, Proof of the main theorem ..... 275
Duality between type and cotype in K-convex spaces, reflexive subspaces of $L_{1}$
Notes and Remarks ..... 280
14. Spaces with Finite Cotype ..... 283Finite cotype is equivalent to non-uniform containment of $\ell_{\infty}^{n}$ 's
Dvoretzky - Rogers Again ..... 283
Factoring Formal Identities ..... 286
The Main Theorem ..... 289
Cotype versus finite factorization and Orlicz's Theorem, Cotype numbers, Cotype numbers of Lebesgue - Bochner spaces, Proofs
Brunel-Sucheston Affairs ..... 298
Ramsey's Theorem, Brunel-Sucheston Theorem, Invariance under spreading Notes and Remarks ..... 303
15. Weakly Compact Operators on $\mathcal{C}(K)$-Spaces ..... 309
Characterization of Weakly Compact Operators ..... 309
An Approximation Scheme ..... 311
Approximation by $p$-summing operators, Characterizations and properties Ultrapower Stability ..... 313
Property (H), Type and Cotype, Rosenthal's Theorem on reflexive subspaces of $L_{1}$ Spaces Verifying Grothendieck's Theorem ..... 316
Subspaces of $\mathcal{C}(K)$ leading to reflexive quotients, Kisliakov's Lemma, Finite cotype is a three space property, Grothendieck's Theorem for quotients of $L_{1}$ by a reflexive subspace
Notes and Remarks ..... 318
16. Type and Cotype in Banach Lattices ..... 326
Functional Calculus ..... 326
Abstract M-spaces, Kakutani's Representation Theorem, Khinchin's Inequality in Banach lattices, Complexification of a Banach lattices( $q, p$ )-Concave Operators330
Definition, Characterization via ( $q, 1$ )-summing operators, Cotype of a Banach lattice, Maurey - Khinchin InequalityThe Rôle of Disjointness333
( $q, 1$ )-summing operators on $\mathcal{C}(K)$ via disjointly supported functions, Cotype $q$ ( $2<\boldsymbol{q}<\infty$ ) of a Banach lattice is determined on disjoint vectors, Maurey - Khinchin Inequality and finite cotype, Order bounded and almost summing operatorsType and Convexity340
( $p, q$ )-convex operators, Duality with $\left(p^{*}, q^{*}\right)$-concavity, Type of Banach lattices Notes and Remarks ..... 341
17. Local Unconditionality ..... 344
Unconditional basis, Unconditional basis constant
Local Unconditional Structure ..... 344
Definitions, Banach lattices have l.u.st., $X$ has l.u.st. iff $X^{* *}$ is complemented in a Banach lattice, l.u.st. and duality
The Gordon-Lewis Inequality ..... 349
GL-Spaces ..... 350
Definitions and duality
GL-Spaces and Cotype ..... 352
GL-spaces of cotype 2, Duality of type and cotype in GL-spaces $\Lambda(2)$-SETS355
Banach Spaces Failing (GL) ..... 358
Failure of (GL) in $\mathcal{L}\left(\ell_{2}\right)$ and in Schatten - von Neumannn classes $\mathcal{S}_{p}, p \neq 2$
Notes and Remarks ..... 363
18. Summing Algebras ..... 373
$p$-Summing Algebras ..... 373
Definitions, Elementary properties and examples ..... 375
Polynomials, Norms of polynomials, Symmetric multilinear forms, Quotients of $p$-summing algebrasQ-Algebras and Operator Algebras378
Q-Algebras and quotients of 1-summing algebras, Quotient algebras of uniformalgebras are operator algebras
A Commutative Non-Operator Algebra ..... 381
The Wiener algebra is not an operator algebra
Failure of the Many Variable von Neumann Inequality ..... 383
A Commutative Non-Q Operator Algebra ..... 386
2-Summing Algebras and Operator Algebras ..... 387
2-summing algebras are operator algebras
STRictly $p$-Summing Algebras ..... 390
Strictly $p$-summing algebras are uniform algebras
Notes and Remarks ..... 393
19. Dvoretzky's Theorem and Factorization of Operators ..... 396
Dvoretzky's Theorem, Factorizing Hilbert - Schmidt operators, Characterization of K-convexity
Fréchet Derivatives of Convex Functions ..... 397
Group Actions and Invariant Measures ..... 400
Actions of the Orthogonal Group ..... 403
Action on spheres, on Grassmannians
Proof of Dvoretzky's Theorem ..... 406
Dvoretzky - Rogers norms, Proof of the theorem, Cotype of spaces of compact operators, Type of spaces of nuclear operators
Basic Sequences ..... 410
Special blocking of basic sequences ..... :
Factorization ..... 412
Factorization of compact Hilbert space operators through subspaces of arbitrary Banach spaces, Application to Hilbert - Schmidt operators, Generalizations Complementation ..... 416Cotype 2 numbers and $\gamma$-summing norm, Existence of finite rank projections ofnice norms
K-Convexity ..... 420
Geodesic Metric, Isoperimetric Inequality, Lévy's Lemma, $X$ is K-convex iff itcontains the $\ell_{2}^{n}$ 's uniformly and uniformly complemented
The Isoperimetric Inequality ..... 424Caps, Qualitative version of the Isoperimetric Inequality, Blaschke's SelectionTheorem, Spherical symmetrization
Notes and Remarks ..... 431
References ..... 435
Author Index ..... 465
Subject Index ..... 469
