

Dominique Bakry • Ivan Gentil • Michel Ledoux

Analysis and Geometry of Markov Diffusion Operators

 Springer

Contents

Part I Markov Semigroups, Basics and Examples

1	Markov Semigroups	3
1.1	Markov Processes and Associated Semigroups	7
1.2	Markov Semigroups, Invariant Measures and Kernels	9
1.3	Chapman-Kolmogorov Equations	16
1.4	Infinitesimal Generators and Carré du Champ Operators	18
1.5	Fokker-Planck Equations	23
1.6	Symmetric Markov Semigroups	24
1.7	Dirichlet Forms and Spectral Decompositions	29
1.8	Ergodicity	32
1.9	Markov Chains	33
1.10	Stochastic Differential Equations and Diffusion Processes	38
1.11	Diffusion Semigroups and Operators	42
1.12	Ellipticity and Hypo-ellipticity	49
1.13	Domains	52
1.14	Summary of Hypotheses (Markov Semigroup)	53
1.15	Working with Markov Semigroups	56
1.16	Curvature-Dimension Condition	70
1.17	Notes and References	74
2	Model Examples	77
2.1	Euclidean Heat Semigroup	78
2.2	Spherical Heat Semigroup	81
2.3	Hyperbolic Heat Semigroup	88
2.4	The Heat Semigroup on a Half-Line and the Bessel Semigroup	92
2.5	The Heat Semigroup on the Circle and on a Bounded Interval	96
2.6	Sturm-Liouville Semigroups on an Interval	97
2.7	Diffusion Semigroups Associated with Orthogonal Polynomials	102
2.8	Notes and References	118

3	Symmetric Markov Diffusion Operators	119
3.1	Markov Triples	120
3.2	Second Order Differential Operators on a Manifold	137
3.3	Heart of Darkness	151
3.4	Summary of Hypotheses (Markov Triple)	168
3.5	Notes and References	173
Part II Three Model Functional Inequalities		
4	Poincaré Inequalities	177
4.1	The Example of the Ornstein-Uhlenbeck Semigroup	178
4.2	Poincaré Inequalities	181
4.3	Tensorization of Poincaré Inequalities	185
4.4	The Example of the Exponential Measure, and Exponential Integrability	187
4.5	Poincaré Inequalities on the Real Line	193
4.6	The Lyapunov Function Method	201
4.7	Local Poincaré Inequalities	206
4.8	Poincaré Inequalities Under a Curvature-Dimension Condition	211
4.9	Brascamp-Lieb Inequalities	215
4.10	Further Spectral Inequalities	220
4.11	Notes and References	230
5	Logarithmic Sobolev Inequalities	235
5.1	Logarithmic Sobolev Inequalities	236
5.2	Entropy Decay and Hypercontractivity	243
5.3	Integrability of Eigenvectors	250
5.4	Logarithmic Sobolev Inequalities and Exponential Integrability	252
5.5	Local Logarithmic Sobolev Inequalities	257
5.6	Infinite-Dimensional Harnack Inequalities	265
5.7	Logarithmic Sobolev Inequalities Under a Curvature-Dimension Condition	268
5.8	Notes and References	273
6	Sobolev Inequalities	277
6.1	Sobolev Inequalities on the Model Spaces	278
6.2	Sobolev and Related Inequalities	279
6.3	Ultracontractivity and Heat Kernel Bounds	286
6.4	Ultracontractivity and Compact Embeddings	290
6.5	Tensorization of Sobolev Inequalities	291
6.6	Sobolev Inequalities and Lipschitz Functions	293
6.7	Local Sobolev Inequalities	296
6.8	Sobolev Inequalities Under a Curvature-Dimension Condition	305
6.9	Conformal Invariance of Sobolev Inequalities	313
6.10	Gagliardo-Nirenberg Inequalities	323
6.11	Fast Diffusion Equations and Sobolev Inequalities	330
6.12	Notes and References	340

Part III Related Functional, Isoperimetric and Transportation Inequalities

7 Generalized Functional Inequalities 347

7.1 Inequalities Between Entropy and Energy 348

7.2 Off-diagonal Heat Kernel Bounds 355

7.3 Examples 362

7.4 Beyond Nash Inequalities 364

7.5 Weak Poincaré Inequalities 373

7.6 Further Families of Functional Inequalities 382

7.7 Summary for the Model Example μ_α 386

7.8 Notes and References 387

8 Capacity and Isoperimetric-Type Inequalities 391

8.1 Capacity Inequalities and Co-area Formulas 392

8.2 Capacity and Sobolev Inequalities 396

8.3 Capacity and Poincaré and Logarithmic Sobolev Inequalities 399

8.4 Capacity and Further Functional Inequalities 403

8.5 Gaussian Isoperimetric-Type Inequalities Under a Curvature Condition 411

8.6 Harnack Inequalities Revisited 421

8.7 From Concentration to Isoperimetry 425

8.8 Notes and References 429

9 Optimal Transportation and Functional Inequalities 433

9.1 Optimal Transportation 434

9.2 Transportation Cost Inequalities 438

9.3 Transportation Proofs of Functional Inequalities 442

9.4 Hamilton-Jacobi Equations 451

9.5 Hypercontractivity of Solutions of Hamilton-Jacobi Equations 454

9.6 Transportation Cost and Logarithmic Sobolev Inequalities 458

9.7 Heat Flow Contraction in Wasserstein Space 462

9.8 Curvature of Metric Measure Spaces 464

9.9 Notes and References 466

Appendices

Appendix A Semigroups of Bounded Operators on a Banach Space 473

A.1 The Hille-Yosida Theory 473

A.2 Symmetric Operators 475

A.3 Friedrichs Extension of Positive Operators 477

A.4 Spectral Decompositions 478

A.5 Essentially Self-adjoint Operators 481

A.6 Compact and Hilbert-Schmidt Operators 483

A.7 Notes and References 485

Appendix B Elements of Stochastic Calculus	487
B.1 Brownian Motion and Stochastic Integrals	487
B.2 The Itô Formula	491
B.3 Stochastic Differential Equations	493
B.4 Diffusion Processes	495
B.5 Notes and References	498
Appendix C Basic Notions in Differential and Riemannian Geometry	499
C.1 Differentiable Manifolds	500
C.2 Some Elementary Euclidean Geometry	502
C.3 Basic Notions in Riemannian Geometry	504
C.4 Riemannian Distance	509
C.5 The Riemannian Γ and Γ_2 Operators	511
C.6 Curvature-Dimension Conditions	513
C.7 Notes and References	518
Afterword	521
Chicken “Gaston Gérard”	521
Notation and List of Symbols	523
Bibliography	527
Index	547